Fundamental Theorem of Calculus I

Assume $g : [a,b] \to \mathbb{R}$ satisfies

- differentiable on (a,b)
- $g'(x)$ integrable on $[a,b]$

$$\Rightarrow \int_a^b g'(x)\,dx = g(b) - g(a)$$

Proof. Let $\varepsilon > 0$

g' integrable on $[a,b] \Rightarrow \exists$ partition P of $[a,b]$ s.t.

$$\boxed{U(g',P) - L(g',P) < \varepsilon} \quad (1)$$

using mean value theorem for integrals we also showed

$$\boxed{L(g',P) \leq g(b) - g(a) \leq U(g',P)} \quad (2)$$

g' integrable $\Rightarrow$

$$\boxed{L(g',P) \leq \int_a^b g'(x)\,dx \leq U(g',P)} \quad (3)$$

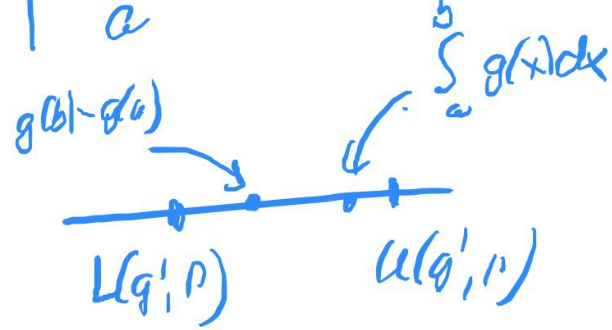
(2) & (3) we see that both $g(b)-g(a)$ and $\int_a^b g'(x)\,dx$

are between $L(g',P)$ and $U(g',P)$

(1) $\Rightarrow$ $U(g',P) - L(g',P) < \varepsilon$

$$\Rightarrow \left| \int_a^b g'(x)\,dx - (g(b)-g(a)) \right| < \varepsilon$$

(geom.



$$\Rightarrow \left| \int_a^b g'(x)\,dx - (g(b)-g(a)) \right| < U(g',P) - L(g',P) < \varepsilon$$

ε arbitrarily small

$$\Rightarrow \int_a^b g'(x)\,dx - (g(b)-g(a)) = 0 \quad \Rightarrow \text{claim.}$$

Applications

① Recall that we directly calculated

$$\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$$

(involved summation formulas etc)

much easier with FTC I:

let $g(x) = \frac{x^{n+1}}{n+1} \Rightarrow g'(x) = x^n$

$$\Rightarrow \int_a^b \underbrace{x^n}_{= g'(x)}\,dx = g(b)-g(a) = \frac{b^{n+1}-a^{n+1}}{n+1}.$$

get result above as special case for $n=2$.

(2) If $g(x) = x^{3/2} \Rightarrow g'(x) = \frac{3}{2} x^{1/2}$

$$\Rightarrow \int_a^b x^{1/2}\, dx = \frac{2}{3} \int_a^b \underbrace{\frac{3}{2} x^{1/2}}_{= g'(x)} dx = \frac{2}{3}\left(g(b) - g(a)\right)$$

$$= \frac{2}{3}\left(b^{3/2} - a^{3/2}\right)$$

Recall from homework problem 33.8:

(If f, g integrable on $[a,b] \Rightarrow fg$ integrable on $[a,b]$)

FTC I

$$\int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)v(x)\,dx =$$

$$= \int_a^b \underbrace{u(x)v'(x) + u'(x)v(x)}_{=g'(x)}\,dx$$

$$\underset{\text{FTC I}}{=} g(b) - g(a)$$

$$= u(b)v(b) - u(a)v(a)$$

Theorem (Integration by Parts)

Assume u, v functions
- differentiable on (a,b)
- u', v' integrable on $[a,b]$

$$\Rightarrow \boxed{\int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)v(x)\,dx = u(b)v(b) - u(a)v(a)}$$

Proof. we use FTC I and product rule for differentiation:

let $g(x) = u(x)v(x)$

$\Rightarrow g'(x) = u(x)v'(x) + u'(x)v(x)$ for all $x \in (a,b)$

by our assumptions $u(x), u'(x), v(x), v'(x)$ all integrable on $[a,b]$

HW 33.9 $\Rightarrow$ $g'(x) = u(x)v'(x) + u'(x)v(x)$ is integrable on $[a,b]$

Fundamental Theorem of Calculus II

Assume $f: [a,b] \to \mathbb{R}$ integrable

$\Rightarrow$ $F(x) = \int_a^x f(t)\,dt$ is a continuous function on $[a,b]$.

moreover, if f continuous at x_0

$\Rightarrow$ F differentiable at x_0 and $F'(x_0) = f(x_0)$

Proof. f integrable $\Rightarrow$ $\exists$ $B>0$ s.t.
$|f(x)| \leq B$ for all $x \in [a,b]$

check our definition!

let $\varepsilon>0$ set $\delta = \frac{\varepsilon}{B}$

if $|x-y| < \delta \Rightarrow$ ② $|F(x) - F(y)| = \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right| =$

$$= \left| \int_y^x f(t)\,dt \right| \leq \int_y^x |f(t)|\,dt$$

assume $y < x$

$$\leq \int_y^x B\,dx$$

$$= |x-y|\,B$$

$$< \frac{\varepsilon}{B} \cdot B = \varepsilon$$

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what about $x < y$?

$$\left| \int_y^x f(t)\,dt \right| = \left| \int_x^y f(t)\,dt \right| \ldots \leq \int_x^y B\,dx = B(y-x)$$

$$\ldots < \varepsilon$$

$\Rightarrow F$ continuous.

assume f cont. at x_0.

Fix $\varepsilon > 0 \Rightarrow \exists\, \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$

whenever $|x - x_0| < \delta$

assume $x < x_0$ (case $x > x_0$ similar, see book)

$$\Rightarrow \boxed{\begin{aligned} F(x_0) - F(x) &= \int_a^{x_0} f(t)\,dt - \int_a^{x} f(t)\,dt \\ &= \int_x^{x_0} f(t)\,dt \end{aligned}} \quad 1$$

moreover:

$$f(x_0) = \frac{1}{x_0 - x} \int_x^{x_0} \underbrace{f(x_0)}_{\text{constant.}}\,dt$$

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right|$$

$$= \left| \frac{1}{x_0 - x} \left(F(x_0) - F(x) - \int_x^{x_0} f(x_0)\, dt \right) \right|$$

$$\overset{①}{=} \frac{1}{x_0 - x} \left(\int_x^{x_0} f(t) - f(x_0)\, dt \right)$$

$$\leq \frac{1}{x_0 - x} \int_x^{x_0} \underbrace{|f(x) - f(x_0)|}_{< \varepsilon \text{ if } |x - x_0| < \delta \text{ by } ②} dt$$

$$\leq \frac{1}{x_0 - x} \int_x^{x_0} \varepsilon\, dt = \frac{1}{\cancel{x_0 - x}} \cdot \varepsilon\, \cancel{(x_0 - x)} = \varepsilon.$$

ε arbitrarily small $\Rightarrow$ claim.